THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

THE one hundred and ninety-first regular meeting of the Society was held in New York City on Saturday, April 28, 1917, extending through the usual morning and afternoon sessions. The attendance included the following twenty-seven members:

Mr. D. R. Belcher, Professor E. G. Bill, Professor E. W. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor W. A. Garrison, Professor O. E. Glenn, Dr. Olive C. Hazlett, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor P. H. Linehan, Professor W. R. Longley, Professor R. L. Moore, Mr. G. W. Mullins, Dr. Alexander Pell, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor Oswald Veblen, Dr. Mary E. Wells, Mr. J. K. Whittemore.

Ex-President E. W. Brown occupied the chair at the morning session, being relieved at the afternoon session by Professor Kasner. The Council announced the election of the following persons to membership in the Society: Professor C. F. F. Garis, Union College; Professor F. J. Holder, University of Pittsburgh; Dr. V. H. Wells, University of Michigan; Professor W. L. Wright, Lincoln University, Pa. Six applications for membership in the Society were received.

Professor L. P. Eisenhart was reelected a member of the Editorial Committee of the *Transactions*, to serve until October 1, 1920. Professor E. R. Hedrick was appointed delegate of the Society to attend the inauguration of President Jessup of the State University of Iowa on May 11–12. A committee consisting of Professors Fite, Birkhoff, and Veblen was appointed to consider and report to the Council any measures which it may be desirable to take to increase the interest and efficiency of the New York meetings of the Society.

Committees were also appointed to consider the question of the legal incorporation of the Society and to prepare a list of nominations of officers and other members of the Council to be elected at the annual meeting in December.

The following papers were read at this meeting:

(1) Professor W. B. FITE: "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivatives."

(2) Dr. SAMUEL BEATTY: "The inversion of an analytic function."

(3) Professor MAURICE FRÉCHET: "Relations entre les notions de limite et de distance."

(4) Professor O. E. GLENN: "A fundamental system of formal covariants mod 2 of the binary cubic."

(5) Professor LUIGI BIANCHI: "Concerning singular transformations B_k of surfaces applicable to quadrics."

(6) Professor J. E. ROWE: "The projection of a line section upon the rational plane cubic curve."

(7) Mr. L. B. ROBINSON: "On partial differential equations which define certain covariants."

(8) Mr. J. K. WHITTEMORE: "Kinematic properties of ruled surfaces."

(9) Dr. OLIVE C. HAZLETT: "On Huntington's set of postulates for abstract geometry."

(10) Mr. E. F. SIMONDS: "Differential invariants in the plane."

(11) Mr. JESSE DOUGLAS: "On certain two-point properties of doubly infinite families of curves on an arbitrary surface."

(12) Professor L. P. EISENHART: "Conjugate planar nets with equal invariants."

(13) Dr. ALEXANDER PELL: "Solutions of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ and their application."

(14) Dr. C. A. FISCHER: "On bilinear and n-linear functionals."

(15) Professor E. B. WILSON: "Classification of real strains in hyperspace."

(16) Professor F. H. SAFFORD: "Irrational transformations of the general elliptic element."

(17) Dr. J. H. WEAVER: "Some algebraic curves."

(18) Professor R. L. MOORE: "A necessary and sufficient condition that a sequence of simple arcs of specified type should be equivalent, from the standpoint of analysis situs, to a sequence of straight segments."

(19) Professor DUNHAM JACKSON: "Second note on the parametric representation of an arbitrary continuous curve."

(20) Professor DUNHAM JACKSON: "Roots and singular points of semi-analytic functions."

(21) Professor OSWALD VEBLEN: "Doubly oriented lines."

(22) Dr. G. M. GREEN: "The intersections of a straight line and a hyperquadric."

(23) Dr. F. W. BEAL: "On a congruence of circles."

(24) Professor G. A. MILLER: "Possible characteristic operators of a group."

(25) Professor R. D. CARMICHAEL: "Examples of a remarkable class of series."

(26) Dr. W. L. HART: "Note on infinite systems of linear equations."

Professor Fréchet's paper was communicated to the Society by Professor D. R. Curtiss, Professor Bianchi's through Professor Eisenhart. Mr. Simonds and Mr. Douglas were introduced by Professor Kasner. Professor Bianchi's paper was read by Professor Eisenhart, and the papers of Dr. Beatty, Professor Fréchet, Professor Rowe, Mr. Robinson, Professor Wilson, Professor Safford, Dr. Weaver, Dr. Green, Dr. Beal, Professor Miller, Professor Carmichael, and Dr. Hart were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Fite's paper deals with differential equations of the second and third orders and with certain ones of general order. The results are analogous to those given in a paper under the same title in the June number of the *Annals of Mathematics*. There may be cited in particular a theorem setting forth conditions under which a solution of an equation of the third order cannot vanish more than twice between two successive roots of another solution.

2. Dr. Beatty's paper appeared in full in the May BULLETIN.

3. Professor Fréchet's paper deals with classes (L) of abstract elements for which the limit of a sequence is defined, and determines conditions to be added such that the class (L) will be a class (D) admitting a definition of distance, but preserving the convergence ideas already adopted. The paper will appear in the *Transactions*.

4. Professor Glenn gives, in this paper, a complete system of covariants mod 2 of a binary cubic form having arbitrary

coefficients. The system is finite. The methods of generation and proof of the completeness of the set of fundamental covariants (which includes also five pure invariants) are developed from the point of view emphasized in the author's paper in volume 17 of the *Transactions* (page 545).

5. A plane and a point on it constitute a *facette*, the point being called the *center*. The tangent planes to a surface and the respective points of contact afford ∞^2 facettes which may be said to constitute the surface. Suppose we have a surface S and with each of its facettes f we assume associated ∞^1 facettes f' in such a way that the center of each facette f' lies in the corresponding plane of f and its plane passes through the center of f. In this way we get ∞^3 facettes f' which ordinarily cannot be coordinated into ∞^1 surfaces S'. It will be all the more unusual if the latter circumstance is satisfied by the ∞^3 facettes f' obtained when S undergoes a deformation by flexure and the facettes f' are carried along in invariable relation with respect to the facettes f. Suppose that for S_0, one deform of S, all the centers of the facettes f' associated with each f lie on the line of intersection of the plane of f and a fixed plane π. Professor Bianchi has proposed and solved the problem: To find what must be assumed concerning S_0 and its relation with the fixed plane π in order that for each deform of S_0 the ∞^3 facettes can be coordinated into ∞^1 surfaces S'. He shows that S_0 must be a quadric; that when it is not a quadric of revolution the fixed plane must be a principal plane and that the planes of the facettes f', associated with each facette f of S_0, envelope a cone projecting from the center of f, the focal conic of S_0 lying in the given diametral plane; when it is a quadric of revolution, the fixed plane must be a meridian plane, and the planes must form two pencils whose axes are the joins of the center of f with the foci of the meridian conic. These geometrical configurations had been obtained by Professor Bianchi formerly in his researches on the transformations of the deforms of quadrics. The paper will be published in the *Transactions*.

6. Professor Rowe's paper appeared in full in the June BULLETIN.

7. Given a differential system

$$A_{11}\frac{\partial f}{\partial x_2}+A_{12}\frac{\partial f}{\partial x_3}+\cdots+A_{1n}\frac{\partial f}{\partial x_{n+1}}=0,$$

$$A_{21}\frac{\partial f}{\partial x_2}+A_{22}\frac{\partial f}{\partial x_3}+\cdots+A_{2n}\frac{\partial f}{\partial x_{n+1}}=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{m1}\frac{\partial f}{\partial x_2}+A_{m2}\frac{\partial f}{\partial x_3}+\cdots+A_{mn}\frac{\partial f}{\partial x_{n+1}}=0,$$

where

A_{i1} are functions of x_1 only,

A_{i2} are functions of x_1, x_2 only,

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

A_{in} are functions of $x_1, x_2, \cdots, x_n$, but not of x_{n+1}

$(i = 1, 2, \cdots, m)$;

Mr. Robinson shows that the singular points of such a system can always be determined in advance and that if the coefficients be polynomials the set can be integrated by a finite number of operations. The equations which define the covariants of systems of ordinary differential equations can always be reduced to such a system. The paper will be published in the next issue of the *Johns Hopkins Circular*.

8. In Mr. Whittemore's paper a study of some properties of ruled surfaces is made by reference of the surface to the moving triedral attached to any curve on the surface. Some properties of the general ruled surface are obtained, the most interesting of which is this: Any ruled surface may be generated by a radius fixed in a sphere whose center moves with unit velocity along the line of striction and which turns about the tangent to the line of striction with angular velocity equal to the reciprocal of the parameter of distribution of the surface; it follows that generators of two ruled surfaces, having the same line of striction and the same parameter of distribution, drawn through a point of the line of striction make a constant angle. Some particularly simple results are obtained for ruled surfaces whose parameters of distribution are equal to the radius of torsion of the line of striction.

9. In 1913, Professor Huntington* gave a set of postulates

* *Math. Annalen*, vol. 73, pp. 522–559.

for ordinary euclidean three-dimensional geometry, in terms of a class K of undefined elements, called "spheres," and the relation R of inclusion. At the time, he thought that the "general laws" were independent; but more recently, he noticed that Postulates 7 and 16 are redundant, and raised the question as to whether any others might be redundant.

Dr. Hazlett's paper shows that several other postulates are redundant, of which may be mentioned Postulates 4 and 6, which are special cases of Postulate 15. Furthermore, if Postulate 15 is satisfied, the vertices of a tetrahedron cannot be coplanar unless two or more coincide, and, in particular, the vertices of a triangle cannot be collinear unless two coincide. A slight change of wording, however, obviates this difficulty. Finally, the triangle transverse axiom (which Professor Huntington proved by the aid of the "general laws" 1–11 and the existence postulates E_1–E_3) is proved without the aid of any postulates beyond the first ten general laws.

10. Mr. Simonds' paper deals with the invariants under groups of point and contact transformations of differential configurations consisting of more than one regular analytic element. Previous papers bearing on this subject are P. Rabut: "Théorie des invariants universels," *Journal de l'Ecole Polytechnique*, 1898; E. Kasner: "Differential elements of the second order, etc.," *American Journal*, 1906, the latter pointing out an error in the former.

The following are the principal results of the present paper:

1. Under a finite continuous group of point transformations there are no *essential* invariants other than those of *two* elements.

2. (*a*) Under an infinite group of point transformations a single regular element has no invariants of order greater than zero. (*b*) There are, however, more general configurations having invariants. (*c*) In the case of certain imprimitive groups the simplest configurations having invariants present some exceptional features. (*d*) If λ_n be the smallest number of elements in the simplest configuration having invariants, $\lambda_n = 2n + 2$ for the entire group, $\lambda_n = n + 3$ for the area-preserving group, $\lambda_n = 3$ for the equilong group. The result for the entire group agrees with Kasner, and not with Rabut. Most of Rabut's results are wrong.

3. A set of theorems on contact transformations corresponding to 1 and 2 are obtained.

11. Mr. Douglas's paper belongs to what may be called "geometry in the neighborhood of a curve," as distinguished from the usual differential geometry which relates chiefly to the neighborhood of a point. Every question in this theory leads analytically to a problem in linear differential equations. This fact is a particular aspect of the principle of Darboux's "auxiliary equation" or Poincaré's "equations of variation," namely, that given an *arbitrary* system of differential equations, the solutions infinitely near to a given one are controlled by a system of *linear* differential equations.

In the present paper, a general doubly infinite curve family F on an arbitrary surface is studied with regard to certain properties involving two points a and b, a considerable distance apart. Let the curve C of the family F join a and b. Suppose at b an element of length $d\sigma$ placed normal to C, and its extremities joined to a by the curves C', C'' of F, these enclosing at a the infinitesimal angle $d\omega$. The ratio $d\sigma/d\omega$ is denoted by $V(a, b)$. When a and b are interchanged in the above construction the value obtained for the ratio $d\sigma/d\omega$ is denoted by $V(b, a)$. A general formula is established which expresses $V(a, b)/V(b, a)$ in terms of a certain integral taken along the curve C from a to b. One of the results flowing from this formula is that the doubly infinite curve families F for which everywhere $V(a, b) = V(b, a)$ are those for which the geodesic curvature is the same, at any given point, for all the curves of the family through that point. This is a generalization of a theorem of Levi-Civita to the effect that the geodesics themselves have the property $V(a, b) = V(b, a)$. A well-known theorem of Straubel on optical families is also generalized and found to hold for a wider category of curve families. The considerations of this paper have been extended to higher dimensions.

12. Koenigs has shown by geometrical considerations that the perspectives of asymptotic lines of a surface from a point on a plane form a conjugate net with equal invariants, and he observed, conversely, that such a planar net is always the projection of asymptotic lines on a surface. He stated that the converse problem is reducible to quadratures. Professor Eisenhart has shown that the methods of ordinary differential equations can be applied to this converse problem, and has found the coordinates of the surface in a form which is a generalization of the formulas of Lelieuvre.

13. In Dr. Pell's paper solutions of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ are obtained which lead to the representation of the coordinates of any curve as the sum of the coordinates of an arbitrary minimal curve and those of a curve lying on the minimal cone, whose arc is equal to the arc of the given curve. The necessary and sufficient conditions that the curve be algebraic are very simple. Some interesting results are easily obtained.

14. It has been proved by Fréchet that a bilinear functional $U[f(s), \varphi(t)]$ can be put in the form

$$\int\int f(s)\varphi(t)d_2u(s, t),$$

where $u(s, t)$ is regular in t, and after modifying the definition of the variation of a function of two variables he has proved that the variation of $u(s, t)$ is the least upper bound of the expression $|U[f, \varphi]|/mf\, m\varphi$, where mf and $m\varphi$ are the maxima of $|f(s)|$ and $|\varphi(t)|$. In the present paper Dr. Fischer has obtained the function $u(s, t)$ in a different way and proved it to be regular in both arguments and unique. Then the work can be extended by mathematical induction to n-linear functionals. In the last part of the paper it is proved that a homogeneous functional of the nth order can always be expressed as a multiple Stieltjes integral.

15. Professor Wilson shows that the algebraic method of Gibbs, which was used in an earlier communication (*Transactions of the Connecticut Academy of Sciences*, volume 14 (1907), pages 1–57) to classify dyadics or strains in hyperspace without regard to reality, affords a very easy means of carrying out the further classification with respect to reality. In addition to the tonics, the shears, and the tonic-shears, we find cyclotonics and cyclotonic-shears. The article will be printed in the *Journal of the Washington Academy of Sciences*.

16. Professor Safford continues in this paper a series of articles which have appeared in the BULLETIN and in the *Archiv der Mathematik und Physik*. These investigations are based upon a formula, published by G. G. A. Biermann and derived from Weierstrass's lectures, which expresses the solution of $[F'(x)]^2 = AF^4(x) + 4BF^3(x) + 6CF^2(x) + 4B'F(x) + A$ as an irrational function of $P(x)$.

17. If two lines l_1 and l_2 in the same plane pass through two fixed points, A and B, and rotate about A and B according to a definite law, the locus of the intersection of l_1 and l_2 will be a definite curve.

Dr. Weaver, by this means, has set up several curves and proved some of the relations existing between the fixed points and the curves.

In particular he has set up a curve C_n of degree n, which has at one of the fixed points an $(n-1)$ point, and such that the polar of the other fixed point with respect to C_n gives C_{n-1}.

18. Suppose that: (*a*) k and l are two parallel lines, (*b*) $A_1, A_2, A_3, \cdots$ and $C_1, C_2, C_3, \cdots$ are two sequences of points lying on k and l respectively and having, as sequential limit points, the points A and B respectively, (*c*) $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3, \cdots$, and ABC are simple continuous arcs no two of which have any point in common and all of which lie, except for their end points, entirely between k and l, (*d*) as $n \doteq \infty$ the arc $A_nB_nC_n$ approaches the arc ABC uniformly as its limit in the sense that if $\epsilon > 0$ there exists n_ϵ such that if $n > n_\epsilon$ then each point of $A_nB_nC_n$ is at a distance less than ϵ from some point of ABC.

Professor Moore proposes to show that a necessary and sufficient condition that such a set of arcs $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3, \cdots$ may be equivalent from the standpoint of analysis situs to a set of parallel segments of straight lines is that for every positive ϵ there should exist a positive δ_ϵ such that if n is any positive integer and X and Y are points of the arc $A_nB_nC_n$ and the distance from X to Y is less than δ_ϵ then that part of the arc $A_nB_nC_n$ which lies between X and Y lies entirely within some circle of radius less than ϵ.

19. In a paper recently presented to the Society (see BULLETIN, volume 23, page 68), Professor Jackson gave a proof of the theorem, apparently due originally to Fréchet, that an arbitrary continuous curve can be represented parametrically in such a form that the coordinates do not remain simultaneously constant throughout any interval of values of the independent variable. In the present note he points out that the theorem is practically obvious in the case of a rectifiable curve, and, by means of a generalization of the process which defines the length of such a curve, obtains a considerably simplified proof for the general case.

20. In his second paper, Professor Jackson shows how certain theorems about roots and singular points of analytic functions of two or more complex variables, Weierstrass's theorem of factorization, for example, can be extended, with appropriate modifications, to functions which are analytic in one of their arguments and merely continuous in all together. The treatment is based primarily on the use of contour integrals.

21. A doubly oriented line is a line in a three-space associated with one sense class among its points and one sense class among its planes. Professor Veblen's note deals with a number of theorems on order relations which cluster about this notion. It will be published in the second volume of Veblen and Young's Projective Geometry.

22. In Dr. Green's note, a simple proof is given of a formula for the intersections of a straight line and a hyperquadric which Professor Coolidge derived by an entirely different method in a paper communicated to the Society at the last annual meeting.

23. Dr. Beal's paper is concerned with a congruence C of circles which satisfies the following conditions. Any circle of the congruence lies in the tangent plane of a surface S and its center is the point of tangency M of the plane with S. At every point P of each circle a line is drawn which makes an angle φ with the plane of the circle. The projection of this line on the plane of the circle makes an angle ψ with the tangent to the circle at P. For any displacements of S in the neighborhood of M the point P is to move at right angles to this line. The surface on which P lies is a surface S_1 of a single parameter family of surfaces and is called a transform of S. The radius R of the circle and the angles φ and ψ are functions of u and v, the curvature coordinates of S. If the three quantities R, φ, and ψ are not restricted, congruences C exist for any surface S. The necessary and sufficient condition that lines of curvature correspond on the surface S and the surface S_1 is that R be constant. When ψ is a constant the principal radii of curvature t_1 and t_2 of any surface S_1 satisfy a very simple relation free from the angle θ which determines a particular surface S_1. When φ is a constant any surface parallel to S has associated with it a congruence C for which

the surfaces S_1 are parallel to the transforms S_1 of S. When R, φ, and ψ, are constants the surfaces S and S_1 are surfaces parallel to a pseudospherical surface and its Backlund transforms.

24. A characteristic operator of a group G is an operator which corresponds to itself in every possible automorphism of G. Every group contains at least one characteristic operator, viz., identity. A necessary and sufficient condition that an abelian group involves another characteristic operator is that it contains one and only one operator of order 2. In particular, an abelian group can never contain more than two characteristic operators. The main object of Professor Miller's paper is to prove the following theorem: It is possible to construct non-abelian groups whose characteristic operators constitute an arbitrary abelian group.

25. Professor Carmichael's paper appeared in full in the June BULLETIN.

26. Dr. Hart considers the infinite system of linear equations

$$\sum_{j=1}^{\infty} a_{ij}x_j = x_i' \qquad (i = 1, 2, \cdots), \tag{1}$$

where $\Sigma_{i=1}^{\infty}|x_i|^2$ and $\Sigma_{i=1}^{\infty}|x_i'|^2$ exist and where the infinite matrix $A = (a_{ij})_{i,j=1,2,\ldots}$ is limited. A condition is given under which (1) acts like a finite system of n equations in n unknowns; that is, a condition under which there exists an integer $w \geqq 0$ such that, for all $(x_1', x_2', \cdots)$ satisfying certain w linear relations, (1) possesses a solution depending linearly on w parameters. The condition is suggested by a property possessed by (1) when $A = E - B$ (E the unit matrix), where B is completely continuous in the Hilbert sense.

F. N. COLE,
Secretary.

THE MODULAR DIFFERENCE OF CLASSES.

BY PROFESSOR P. J. DANIELL.

In the general analysis of numerical functions the modular difference of two numbers is an extremely valuable concept. This paper discusses an analogous concept in the calculus of classes. These classes may be logical classes in general or may be point sets. The following notation* is used in which capital letters denote classes.

AB = logical product of A and B.

$A + B$ = logical sum of A and B.

$\overline{A}$ = complement of A or CA in Borel's notation.

$A < B$ means A is included in B.

$A > B$ means A includes B.

$A - B$ is defined only when $A > B$ and is then equal to $A\overline{B}$. The modular difference of two numbers a, b satisfies the two conditions

(1) when $a \geqq b$, $| a - b | = a - b$,

(2) $| a - b |$ is symmetrical in a and b.

By analogy the modular difference of two classes A, B denoted by $| A - B |$ should satisfy

(1) when $A > B$, $| A - B | = A\overline{B} = A - B$,

(2) $| A - B |$ is symmetrical in A and B.

Definition.—The modular difference of two classes is the logical sum of the logical product of the first into the complement of the second and the logical product of the second into the complement of the first.

$$| A - B | = \overline{A}B + A\overline{B} \qquad \text{(Def.)}.$$

The reader is to note that the modular difference is defined even in cases where the difference according to Borel is not defined. That is to say, the modular difference is a direct operation on the two classes, not an operation on their difference.

Using the definition above and the analysis of symbolic logic the following properties can be proved.

* Borel, Théorie des Fonctions des Variables réelles, p. 16. Baldwin, Dictionary of Philosophy and Psychology, article "Symbolic Logic." Whitehead and Russell, Principia Mathematica, chapter on "Calculus of Classes."

Properties.

1(1). $|A-B| = A\overline{B}+\overline{A}B = (A+B)(\overline{A}+\overline{B}) = |\overline{A}-\overline{B}|.$

1(2). $|\overline{A-B}| = AB+\overline{A}\overline{B} = |\overline{A}-B| = |A-\overline{B}|.$

1(3). If $|A-B| = 0$, $A = B$ and vice versa.

(In this, 0 may be regarded as a set or class with no elements.)

2(1). $||A-B|-C| = |A-|B-C||.$

2(1) cor. If $|A-B| = C$, $A = |B-C|$.

2(2). $||A-B|-|B-C|| = |A-C|.$

2(2) cor. If $|A-B| = |B-C|$, $A = C$ and vice versa.

2(3). $|A-B||B-C||C-A| = 0.$

2(4). $C|A-B| = |CA-CB|.$

3(1). $|(A+B)-C| = |A-C||B-C|+|A-B|\overline{C}.$

3(2). $|AB-C| = |A-C||B-C|+|A-B|C.$

3(3). $|A-C|+|B-C| = |(A+B)-C|+|A-B|C.$

3(4). $|A-C|+|B-C| = |AB-C|+|A-B|\overline{C}.$

4(1). $|A-B| < A+B.$

4(2). $|A-C| < |A-B|+|B-C|.$

4(3). $|A-C||B-C| < |(A+B)-C| < |A-C| + |B-C|.$

4(4). $|A-C||B-C| < |AB-C| < |A-C|+|B-C|.$

5. $(A+B+C+\cdots+N)-ABC\cdots N = |A-B|+|A-C|+\cdots+|A-N|.$

The standard type of equation in symbolic logic having a unique solution is of the form

$$X\overline{A}+\overline{X}A = B.$$

Using our notation, this becomes $|X-A| = B$ or by 2(1) cor. the solution is $X = |B-A| = B\overline{A}+\overline{B}A$.

The properties under 4 are modular properties which are immediate deductions from the equations under 1(1), 2(2) and 3. Property 5 will be useful to us in what follows.

Definition.—A sequence of classes $S_1, S_2, \cdots, S_n, \cdots$ is said to be a standard sequence, or S-sequence, if $S_n > S_{n+1}$ for all values of n and if $\lim S_n = 0$, that is, if each class includes the next and if there is no element common to all.

A sequence is said to be decreasing if each class includes the next and to be increasing if each class is included in the next.

In this paper inclusion denotes logical inclusion, that is to say, equality is also inclusion in the sense that if $A = B$, then $A > B$ and $B > A$.

Limit.—A sequence of classes $A_1, A_2, \cdots, A_n, \cdots$ is said

to possess a limiting class A if an A can be found such that $|A - A_n| < S_n$ for all values of n, where S_n is the nth member of a standard sequence. We proceed to prove that this definition coincides in content with that given by Borel (see reference above).

Firstly, if the sequence is decreasing and A is its limit by Borel,

$$|A - A_n| = A_n - A < A_n - A$$

and $A_n - A$ forms an S-sequence. Secondly, if the sequence is increasing and A is its limit by Borel,

$$|A - A_n| = A - A_n < A - A_n$$

and $A - A_n$ forms an S-sequence. Thirdly, if in general the sequence possesses a Borel limit A,

$$|(A_m + A_{m+1} + \cdots + A_{m+p}) - A_n|$$
$$< |A_n - A_m| + |A_n - A_{m+1}| + \cdots + |A_n - A_{m+p}| \qquad \text{[Prop. 4(3)]}$$
$$< |A_n - A_{n+1}| + |A_n - A_{n+2}| + \cdots + |A_n - A_{m+p}| \qquad \text{if } m > n$$
$$< (A_n + A_{n+1} + \cdots + A_{m+p}) - A_n A_{n+1} \cdots A_{m+p} \qquad \text{[Prop. 5]}$$
$$< (A_n + A_{n+1} + \cdots) - A_n A_{n+1} \cdots,$$
$$|(A_m + A_{m+1} + \cdots) - A_n|$$
$$< (A_n + A_{n+1} + \cdots) - A_n A_{n+1} \cdots \qquad \text{if } m > n.$$

But the right-hand side forms a decreasing sequence and has the limit 0 if the limit of sequence A_n exists according to Borel. Thus the right-hand side is an S_n of an S-sequence. $|(A_m + A_{m+1} + \cdots) - A_n| < S_n$ for all values of $m > n$. Hence in the limit as m increases indefinitely

$$|\text{Borel limit } A - A_n| < S_n.$$

Hence the sequence A_n has the Borel limit A, according to our definition, whenever it has a limit by Borel's definition. Again, suppose that it has a limit A by our definition

$$|A - A_n| < S_n$$

of some S sequence.

$$\begin{aligned}
&|\,(A_n + A_{n+1} + \cdots + A_{n+p}) - A\,| \\
&\quad < |\,A - A_n\,| + |\,A - A_{n+1}\,| + \cdots + |\,A - A_{n+p}\,| \\
&\qquad\qquad\qquad

If, on the other hand,

$$|A_n - A_{n+p}| < S_n$$

$$\begin{aligned}(A_n + A_{n+1} + \cdots + A_{n+p}) - A_nA_{n+1}\cdots A_{n+p} \\ = |A_n - A_{n+1}| + |A_n - A_{n+2}| + \cdots + |A_n - A_{n+p}| \\ < S_n + S_n + \cdots + S_n \\ < S_n.\end{aligned}$$

In the limit when p increases indefinitely

$$(A_n + A_{n+1} + \cdots) - A_nA_{n+1}\cdots < S_n.$$

The left member is a sequence decreasing with n and it must have the limit 0 for S_n has the limit 0. Consequently the sequence A_n has a limit according to Borel.

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ON THE PRINCIPAL UNITS OF AN ALGEBRAIC DOMAIN $k(\mathfrak{p}, \alpha)$.

BY DR. G. E. WAHLIN.

(Read before the American Mathematical Society, April 10, 1914.)

Introduction.

THE following paper is the result of an investigation of a problem connected with the representation of the algebraic numbers in the form $\pi^a\omega^\beta e^\gamma$.*

Throughout the discussion I shall use the following notation. By p I mean a rational prime and by $\mathfrak{p}$ any prime divisor of p. f is the degree of $\mathfrak{p}$, i. e., $N(\mathfrak{p}) = p^f$ and $\mathfrak{p}^\sigma$ is the highest power of $\mathfrak{p}$ contained in p. By π I mean a prime number of the domain $k(\mathfrak{p}, \alpha)$, where α is an arbitrary algebraic number. The numbers of $k(\mathfrak{p}, \alpha)$ are then of the form $a_\rho\pi^\rho + a_{\rho+1}\pi^{\rho+1} + \cdots$. A number in which $\rho = 0$ and a_ρ is relatively prime to $\mathfrak{p}$ is called a unit and in particular if $a_\rho = 1$ it is called a principal unit.

* Hensel, *Crelle's Journal*, vol. 145.

The Equation $x^{p^n} - E = 0(\mathfrak{p})$.

For the present we shall let E be any unit of $k(\mathfrak{p}, \alpha)$. From the general theory of algebraic numbers* we know that there exists a certain rational integer μ such that the equation

$$x^{p^n} - E = 0(\mathfrak{p}) \tag{1}$$

has a solution in $k(\mathfrak{p}, \alpha)$ if the congruence

$$x^{p^n} - E \equiv 0 \bmod \mathfrak{p}^{\mu+1} \tag{2}$$

has a solution in this domain. The present section is devoted to the computation of the value of μ.

This determination of μ can be accomplished by making use of a known theorem.†

Since E is a unit, it follows that any solution E_1 of (2) is also relatively prime to $\mathfrak{p}$. Therefore if we put $F(x) = x^{p^n} - E$ and denote its ith derivative by $F^{(i)}(x)$ we see that the order of

$$F^{(i)}(E_1)/i! = \frac{p^n(p^n-1)\cdots(p^n-i+1)}{i!} E_1^{p^n-i}$$

is the same as the order of $C^{(i)} = p^n!/i!(p^n - i)!$.

The order of $m!$ in $k(p)$ is $(m - S_m)/(p-1)$‡ where S_m is the sum of the coefficients in the reduced p-adic representation of m. Hence since $S_{p^n} = 1$ we know that in $k(p)$ the order of $C^{(i)}$ is

$$\frac{p^n - 1}{p-1} - \frac{i - S_i}{p-1} - \frac{p^n - i - S_{p^n-i}}{p-1} = \frac{S_i + S_{p^n-i} - 1}{p-1}.$$

Let us denote the order of i by ρ and suppose that in its reduced p-adic representation $i = a_\rho p^\rho + a_{\rho+1}p^{\rho+1} + \ldots + a_{n-1}p^{n-1}$. Since $i \leqq p^n$ the representation cannot have a term containing a higher power of p than p^{n-1}, excepting in the case where $i = p^n$ and then the order of $C^{(i)}$ is zero. The number p^n can be written in the form $p \cdot p^\rho + (p-1)p^{\rho+1} + \ldots + (p-1)p^{n-1}$, and hence

$$p^n - i = (p - a_\rho)p^\rho + (p - a_{\rho+1} - 1)p^{\rho+1} + \ldots + (p - a_{n-1} - 1)p^{n-1},$$

* Hensel, Theorie der algebraischen Zahlen, Kap. 4, § 4. (The method there used by Professor Hensel can be extended to any domain.)

† Ibid., Kap. 4, § 4, pp. 72–74.

‡ Ibid., p. 111.

which as is easily seen is also in the reduced form. Hence

$$S_i = a_\rho + a_{\rho+1} + \ldots + a_{n-1}$$

and

$$S_{p^n-i} = p - a_\rho + p - a_{\rho+1} - 1 + \ldots + p - a_{n-1} - 1$$

and

$$S_i + S_{p^n-i} = (n-\rho)p - (n-\rho-1),$$

whence we have

$$(S_i + S_{p^n-i} - 1)/(p-1) = n - \rho.$$

Since $\mathfrak{p}^\sigma$ is the highest power of $\mathfrak{p}$ in p we see now that $\rho^{(i)}$, the order of $C^{(i)}$ in $k(\mathfrak{p}, \alpha)$, is equal to $\sigma(n-\rho)$.

If we now form the expression $(i\rho' - \rho^{(i)})/(i-1)$* we see that this is equal to

$$\sigma\frac{ni - (n+\rho)}{i-1} = \sigma\left(n + \frac{\rho}{i-1}\right)$$

since $\rho' = n\sigma$. The value of μ sought is the largest integer which is less than or equal to

$$\max \sigma\left(n + \frac{\rho}{i-1}\right) \text{ for } i = 2, 3, \ldots, p^n.$$

Since n and σ are independent of i, it is evident that this maximum occurs when $\rho/(i-1)$ is maximum and we shall therefore determine the value of i for which such is the case.

If we first consider the values of i of a given order ρ it is clear that $\rho/(i-1)$ is maximum when i is minimum and hence when $i = p^\rho$ and the maximum value of $\rho/(p^\rho - 1)$ as ρ varies over the numbers 1, 2, ... n is therefore the same as the maximum value of $\rho/(i-1)$ as i varies over the numbers 2, 3, ... p^n. We note here that for $1 < i < p$, $\rho = 0$ and $\rho/(i-1) = 0$.

Let us now turn our attention to the expression

$$\psi(\rho) = \rho/(p^\rho - 1).$$

Differentiating, we have

$$\psi'(\rho) = \frac{p^\rho - 1 - \rho p^\rho \log p}{(p^\rho-1)^2} = \frac{p^\rho(1 - \rho \log p) - 1}{(p^\rho-1)^2}.$$

* Hensel, Theorie der algebraischen Zahlen, Kap. 4, § 4.

If $p > 2$, $\log p > 1$ and hence, since $\rho \geqq 1$, $\psi'(\rho) < 0$. The function $\psi(\rho)$ is therefore a decreasing function for $\rho \geqq 1$ and the maximum value in the required interval therefore occurs when $\rho = 1$. This maximum value is $1/(p-1) > 0$ and since for $1 < i < p$, $\rho/(i-1) = 0$, $1/(p-1)$ is the maximum value of $\rho/(i-1)$. If $p = 2 < e < 4$, since $e^{1/2} < 2$ we have $\frac{1}{2} < \log 2 < 1$ and hence for $\rho \geqq 2$ we have $\rho \log 2 > 1$ and as before $\psi'(\rho) < 0$. Therefore for $\rho \geqq 2$, $\psi(\rho)$ is decreasing and must be maximum, in the given interval, when $\rho = 2$. Hence when ρ takes the values $1, 2, \ldots, n$, $\psi(\rho)$ must be maximum either at $\rho = 1$ or $\rho = 2$.

$$\text{For } \rho = 1, \psi(\rho) = \frac{1}{2-1} = 1.$$

$$\text{For } \rho = 2, \psi(\rho) = \frac{2}{4-1} = \tfrac{2}{3}$$

and hence, as in the preceding case, the maximum value occurs when $\rho = 1$ and again the maximum is $1/(p-1)$. Therefore

$$\max\left(\frac{i\rho' - \rho^{(i)}}{i-1}\right) = \sigma\left\{n + \frac{1}{p-1}\right\}$$

and if we put $k = [\sigma/(p-1)]$ we have

$$\mu = n\sigma + k.$$

A Certain Residue Group in $k(\mathfrak{p}, \alpha)$.

We shall suppose that the domain $k(\mathfrak{p}, \alpha)$ contains all the p^rth roots of unity while no primitive p^{r+1}th root of unity is contained in it. We shall in this discussion need the number μ of the preceding section for the special case when $n = r + 1$ and shall therefore put $\mu = r\sigma + \sigma + k \geqq 1 + k$.

Every principal unit E of our domain is, modulo $p^{\mu+1}$, congruent to one and only one of the $p^{\mu f}$ units $1 + a_1\pi + a_2\pi^2 + \cdots + a_\mu\pi^\mu$ where the a_i vary independently over the p^f numbers of a complete residual system modulo $\mathfrak{p}$. Since the product and quotient of two principal units are principal units it is evident that these residues and hence the E's themselves form an abelian group of order $p^{\mu f}$ with respect to the modulus $p^{\mu+1}$. This group we shall denote by G. Since G is an abelian group we know that it is the product of cyclic groups. These cyclic groups we shall denote by $C_1, C_2, \cdots C_h$, and the order of C_i we shall denote by p^{r_i}. (The order must

be a power of p since it is a divisor of $p^{\mu f}$.) We shall moreover assume that $r_1 \geqq r_2 \geqq r_3 \geqq \cdots \geqq r_h$.

Let m be that one of the numbers $1, 2, \cdots, h$ such that $r_m > r \geqq r_{m+1}$ or if $r_h > r$, $h = m$. We shall first see that r cannot be greater than r_1. G cannot contain an element of period greater than p^{r_1} and hence if R is a primitive p^rth root of unity, it is an element of G and therefore $R^{p^{r_1}} \equiv 1 \bmod p^{\mu+1}$ and hence also modulo p^{k+1}, since $\mu \geqq k+1$. But since $R^{p^{r_1}} \equiv 1 \bmod p^{k+1}$ it is an exponential unit* and we can therefore write $R^{p^{r_1}} = e^{\nu}(\mathfrak{p})$. By raising both members of this equation to the power p^{r-r_1} we have $e^{\nu p^{r-r_1}} = 1(\mathfrak{p})$ and hence $\nu p^{r-r_1} = 0\ (\mathfrak{p})$ and $\nu = 0\ (\mathfrak{p})$. But then $R^{p^{r_1}} = e^{\nu} = 1(\mathfrak{p})$ and since R is a primitive p^rth root of unity this is impossible unless $r \leqq r_1$.

In the same way it follows that for $t < r$, $R^{p^t} \not\equiv 1 \bmod \mathfrak{p}^{\mu+1}$ and hence R and its powers form a cyclic subgroup of G, of order p^r.

If $r = r_1$ it is evident, from the proof of the theorem, that every abelian group can be written as the product of cyclic subgroups,† that we can put $C_1 = C$ where C is the cyclic group generated by R. If however $m > 1$ we shall next see that no power of R excepting R^{p^r} is modulo $p^{\mu+1}$ congruent to a number in the product $C_1 \cdot C_2 \cdots C_m$.

Let us denote by E_i any generator of the cyclic group C_i and let us suppose that

$$E_1^{n_1 p^{\lambda_1}} \cdot E_2^{n_2 p^{\lambda_2}} \cdots E_m^{n_m p^{\lambda_m}} \equiv R^{np^{\lambda}} \bmod \mathfrak{p}^{\mu+1}, \tag{3}$$

where we assume that $n, n_1, n_2, \cdots, n_m$ are rational integers relatively prime to p and $0 \leqq \lambda < r$ and $0 \leqq \lambda_i < r_i$ $(i = 1, 2, \cdots m)$. By raising both members of (3) to the power $p^{r-\lambda}$ we have

$$E_1^{n_1 p^{\lambda_1 + r - \lambda}} \cdot E_2^{n_2 p^{\lambda_2 + r - \lambda}} \cdots E_m^{n_m p^{\lambda_m + r - \lambda}} \equiv 1 \bmod \mathfrak{p}^{\mu+1} \tag{4}$$

and from the fact that G is an abelian group and $C_1, C_2, \cdots, C_h$ the base we know that this is possible when and only when the exponent of each E_i is divisible by p^{r_i}. Hence $\lambda_i + r - \lambda \geqq r_i$ and since for $i \leqq m$, $r_i > r$, we have $\lambda_i \geqq r_i - r + \lambda > \lambda$. If we now let $l = \min(\lambda_1, \lambda_2, \cdots \lambda_m)$ and put

$$E = E_1^{n_1 p^{\lambda_1 - l}} \cdot E_2^{n_2 p^{\lambda_2 - l}} \cdots E_m^{n_m p^{\lambda_m - l}}$$

* Hensel, *Crelle's Journal*, vol. 145, pp. 94–95.
† Weber, Algebra, vol. II, pp. 3, 38–45.

we can write (3) in the form

$$E^{p^l} \equiv R^{np^\lambda} \bmod \mathfrak{p}^{\mu+1}.$$

Since $\lambda_i > \lambda$ it follows that $l > \lambda$.

If we now put $t =$ minimum $(l, r+1)$ and use the result of the first part of this paper we can from the last congruence conclude that the equation

$$x^{p^t} = R^{np^\lambda} (\mathfrak{p}) \tag{5}$$

has a solution in $k(\mathfrak{p}, \alpha)$. Let us denote this solution by $\mathfrak{A}$. Then $\mathfrak{A}^{p^{t+r-\lambda}} = 1(\mathfrak{p})$. Since R is a primitive p^rth root of unity and n is relatively prime to p, R^n is also a primitive p^rth root of unity and hence

$$\mathfrak{A}^{p^{t+r-\lambda-1}} = (R^n)^{p^{r-1}} \not\equiv 1(\mathfrak{p}).$$

$\mathfrak{A}$ is therefore a primitive $p^{t+r-\lambda}$th root of unity which is contained in $k(\mathfrak{p}, \alpha)$.

But we have seen that $l > \lambda$ and have assumed that $\lambda < r$ and hence $r + 1 > \lambda$ and consequently $t =$ min $(r+1, l) > \lambda$ and $t + r - \lambda > r$. But this contradicts our assumption that $k(\mathfrak{p}, \alpha)$ contains no primitive p^{r+1}th root of unity.

Hence (3) is impossible when $\lambda < r$ and hence no power of R excepting $R^{p^r} = R^0$ or power of R^{p^r} can be congruent, modulo $p^{\mu+1}$ to the left hand member of (3).

From this it now follows that in the construction of the base of G we can put $C_{m+1} = C$ and hence have

$$G = C_1 \cdot C_2 \cdots C_m \cdot C \cdot C_{m+2} \cdots C_h.$$

If we put $G_1 = C_1 \cdot C_2 \cdots C_m \cdot C_{m+2} \cdots C_h$, this is also an abelian group and

$$G = G_1 \cdot C. \tag{6}$$

The result may now be summed up in the following

THEOREM: *If the domain $k(\mathfrak{p}, \alpha)$ contains a primitive p^rth root of unity but no primitive p^{r+1}th root of unity, and if we denote by μ the number $r\sigma + \sigma + k$ where σ is the exponent of the prime divisor $\mathfrak{p}$ in p and $k = [\sigma/(p-1)]$, then the abelian group consisting of the principal units of $k(\mathfrak{p}, \alpha)$ modulo $\mathfrak{p}^{\mu+1}$ is the product of an abelian group G_1 and the cyclic group C whose elements are the p^rth roots of unity.*

UNIVERSITY OF ILLINOIS.

PROJECTIVE GEOMETRY.

Synthetic Projective Geometry. By DERRICK NORMAN LEHMER. Boston, 1917. 16mo. xiii+120 pp.

Socrates is a lobster.
Lobsters are mortal; therefore
Socrates is mortal.

IT was from some such example as this that we first learned how a correct conclusion could be drawn from a faulty minor premise. The book before us is based upon a similar syllogism.

"The subject of projective geometry is . . . destined soon to force its way into the secondary schools;"

That which is taught in secondary schools needs a clear and very simple text-book; therefore

Projective geometry needs a clear and very simple text-book.

With this conclusion we most heartily agree, we wish that we might also agree with the minor premise. Alas, our ears ring with the cries of those who would abolish geometry from our high schools altogether for the mathematically worded reason that it does not "function." We suppose that space will continue to exist, even after people have discovered that there is no need to study its properties, and perhaps the nice little book before us will have an archæological importance long after its subject matter has been proved valueless by the refined tests of laboratory psychology.

It *is* a nice little book, in spite of a bad start and certain other faults which we shall point out in detail. Utterly conventional in type, with the classic methods of the projective geometry of the nineteenth century everywhere in evidence, it is clear, interesting, and readable; the simplest and most elementary book on the subject that we know. The figures are rather small, but the general page impression is pleasing, and the proof-reading seems flawless.

The book, we say, makes a bad start; let us explain. The first chapter deals with one-to-one correspondence, an abstract notion no matter how carefully explained. No later than page 3 we read: "If a one-to-one correspondence has been set up between the objects of one set and the objects of another set, then the inference may usually be drawn that they have the same number of elements."

"Usually be drawn," we snort, "and when, pray, may it not be drawn?" We are answered at the bottom of the page.

Two lines AB and $A'B'$ are drawn of different lengths, and put into one-to-one correspondence, "but," says the author, "it would be absurd to infer from this that there are as many points on AB as on $A'B'$."

The conclusion that one would naturally draw from this astonishing statement is that the author has never heard of the modern theory of assemblages, but such a conclusion would be erroneous, for we read, page 21: "It is perfectly possible to set up a one-to-one correspondence between the points of a line and those of a plane."

We receive a second jolt on page 8, where we are told that the totality of points of a line form an infinitude of the first order, while those of a plane form an infinitude of the second order. What is meant is, of course, that the points of a line may be made to depend continuously on one parameter, while those of a plane depend continuously on two parameters. Since, however, it is assumed that the reader's infant mind is not up to understanding such a statement, he is left to puzzle out as best he may how two infinitudes of different orders can be put into one-to-one correspondence. We learn on page 6 that a projective transformation is a continuous transformation and on page 11 we receive this warning: "It must not be forgotten, however, that we are considering only continuous correspondences." It is a wise restriction; let us turn to page 7. "It is easy to set up a one-to-one correspondence between the points in a plane, and the system of lines cutting across two lines which lie in different planes." Let us remember that it has never been suggested that a one-to-one correspondence could have exceptional elements, and the plane in question is the projective plane which has the connectivity one. The points on two lines, however, may be put into one-to-one correspondence with the points on a ruled hyperboloid, and the statement is that it is easy to set up a continuous one-to-one correspondence between the points of a surface of connectivity one and one of connectivity two. Finally we have the problem, page 13; remember that this is one of the first problems set to the geometrical infants for whom the book is written.

"Is the axiom 'The whole is greater than one of its parts' applicable to infinite assemblages?"

What answer is expected? If the author asks in the mood displayed on page 11, where he says that there is a one-to-one correspondence between the points of a line, and those of a plane, the wise pupil will answer that the axiom has lost all validity. But if the author is in the mood of page 6 where he says that it would be absurd to say that there were the same number of points on two lines of different lengths, then the prudent course would seem to be to reply that the axiom is still doing business at the old stand.

Let us epitomize these ill-natured remarks. We are not suggesting that pupils for whom this book is intended should be taught the theory of assemblages, or the analysis situs. Heaven defend us from any such idea! What we do insist is that if it be bad teaching to tell them things which they do not understand, it is worse teaching to tell them things which are not true. Certainly in their case "It is better not to know so many things, than to know so many things that ain't so."

In the second chapter we get a real start in the proper subject matter of the book. The topics dealt with are the fundamental principles of projection and intersection. Of course Desargues's two triangle theorem is of first importance here; let us see what proof the author gives. He starts ahead in the conventional way, taking the case where the two triangles are in different planes, and this case is done up in good shape. Then the proof comes to an abrupt end* with these words:

"If, now, we consider a plane figure, the points P, Q, and R still lie on a straight line, which proves the theorem."

What the author doubtless means is that if one plane approach the other as a limit, the line P, Q, R will approach a definite limiting position in that plane, so that the proof still holds. What he says, however, is that these three points are collinear, even when the triangles are coplanar. If this be not self-evident, why not prove it, since it is the crux of the situation? But if it be self-evident, why bother with the three-dimensional case in a book on plane geometry?

After this, the book runs ahead smoothly enough for sixteen pages, the next difficulty occurring at that most dangerous point, the fundamental theorem of projective geometry:

"*If two projective one-dimensional forms have more than two self-corresponding elements, they are identical.*"

* P. 16.

This theorem has had a long and stormy career. The present author knows well that it can not be proved by projective synthetic methods without the aid of continuity. He therefore introduces, *ad hoc*, an axiom to the effect that a projective correspondence is a continuous correspondence. Unfortunately, even with the timely assistance of this assumption, the proof is not perfect, for it is assumed* that if at each stage of a certain process the interval which contains a given point is diminished, then the interval itself can be made as small as we please.

The fourth and fifth chapters deal with point rows and linear pencils of the second order, i. e., point and line conics. Pascal and Brianchon play leading rôles, the whole discussion goes ahead smoothly enough. The same may be said of the following chapter which deals with poles and polars, the only exception being a small mistake which is not unknown in elementary text-books of analytic geometry. The polar of a point is defined in a fashion which is inadmissible when the point lies on the conic.† Consequently the following fundamental theorem whereby if one point lie on the polar of a second, the second is on the polar of the first, suffers an exception when one of these points is on the conic. It is better to define the tangent as the polar of a point of the curve; the fundamental theorem then suffers no exception.

The eighth chapter, which might better have been put in the seventh place, deals with the properties of involutions. The author says in the preface that he has never felt satisfied with the usual treatment of involutions by means of circles and anharmonic ratios; the present treatment represents his idea of how the subject may be made easier and more consistent. His treatment is certainly purely projective, and entirely logical, except for the unproved statement‡ that an involution must have two double points or none. It seems to us, however, that the order of topics is artificial, and in consequence, some of the proofs quite needlessly hard. He begins with point involutions; three pairs of collinear points are said to be in involution if they lie on three pairs of opposite sides of a complete quadrangle. Then comes a proof of Desargues's involution theorem for conics which is quite

* P. 32.
† P. 57.
‡ P. 73.

complicated, and lastly what the author calls the fundamental theorem, namely, that if in two projective fundamental one-dimensional forms, a single pair correspond interchangeably, the same is true of every pair. The proof of this covers a whole page, and even then the last details are left to the reader. Then we have dual definitions and theorems for involutions of lines. Finally we deduce from the fundamental theorem that a transversal will meet an involution of concurrent lines in an involution range of points. All this is, as we said, a perfectly legitimate way to handle the subject; we can not think that it is a particularly easy one, for the proofs of these same theorems in such a book as Veblen and Young's "Projective Geometry"* even when written out "in a language understanded of the people" are demonstrably shorter than those of the present author.

Chapters VII and IX deal with metrical properties of conics and of involutions respectively. Purists will incline to look upon the introduction of this material as a blemish on the beauty or consistency of the structure. Our own view is just the opposite. These metrical ideas are introduced in illustration of special cases of the theorems developed, not in the course of the logical structure. The chief reason why we welcome them is, however, a didactic one. There can be no doubt that the young geometer feels more interested and more at home when he is dealing with metrical theorems, than when he is occupied with exclusively projective ones. The former certainly come more nearly within his ordinary range of interests. Moreover, the great danger is that projective geometry may appear as a subject apart, but slightly connected with any other mathematical branch. This danger is somewhat obviated when the method of approach is algebraic, as the student sees the connection between the geometric theorems and their algebraic formulation. When, however, as in the present instance, the treatment is purely synthetic, the subject is likely to remain entirely hanging in the air, unless it be tied to the earth by being linked up with more familiar metrical material. We therefore welcome these two chapters, IX being especially interesting and valuable.

The last chapter gives a brief résumé of the history of projective geometry. The author's idea is that the time to learn the history of a subject is after one has found out what the subject is. There is something to be said for this view.

*Vol. 1, Boston, 1910, especially pp. 102, 146.

Now we have exhausted our store of invective. Let us close as we began by saying that it is a clear and interesting little book, whose appearance we heartily welcome.

J. L. COOLIDGE.

NON-EUCLIDEAN GEOMETRY.

The Elements of Non-Euclidean Plane Geometry and Trigonometry. By H. S. CARSLAW. London, Longmans, Green and Company, 1916. 16mo. vii+179 pp.

MORE than a score of years ago a writer of fluent pen published a short article with the title "Is the Non-Euclidean Geometry Inevitable?" His decision was affirmative, his interpretation of the question being "Are the conclusions of non-euclidean geometry inevitable?" We have learnt that they are, but we have also learnt that both the conclusions and the consequences are unavoidable, and that books dealing with the subject must be expected with certainty and met with fortitude. The number already published is already large, as we see by Somerville's compendious bibliography,* and the desire to publish others is so strong that not even the war can choke it. But there is always room at the top, and if a newcomer does not succeed in getting there himself, he raises the existing leader so much the higher. The book before us is a good one. We shall cheerfully damn it in detail later on, let us first praise it in general.

To begin with it is an interesting book. The field covered is not wide. Some things are omitted which we regret to miss, but nothing is included which might better have been left out. The choice of material seems largely guided by the principle that the book was primarily written for teachers, and the author has not shut his eyes to the fact that the average Anglo-Saxon mathematical teacher is too busy to read a long mathematical book, and too weak scientifically to understand a deep one. In consequence of this, certain standard topics like the realization of non-euclidean geometry on surfaces of constant total curvature or the subtleties of the Cayleyan metric are passed over in silence. The most serious omissions come from the restriction to plane geometry. In euclidean

* London, Harrison, 1911.

geometry, the most interesting elementary facts about the geometry of three dimensions may be inferred by the light of nature from what happens in the plane, but in non-euclidean space there are exceptions to this convenient rule. For instance, there is nothing in the geometry of the elliptic plane that foreshadows the properties of Clifford's skew parallel lines in space. These parallels constitute one of the most novel and interesting figures in all of non-euclidean geometry; it is a pity that a reader whose knowledge is bounded by the present book must remain in ignorance of them. It is pleasant to note in this connection that the author has a keen didactic instinct. Whenever he is forced to give a proof that is long or difficult, he divides the work into stages of reasonable length, explaining clearly just what is accomplished in each stage.

The first, yes, and much the greatest difficulty with which the writer on non-euclidean geometry has to deal is that of the fundamental assumptions or axioms. The present writer has met this difficulty in exemplary fashion. He recognizes that it would be equally unwise to give a long analysis of axioms according to modern abstract principles, or merely to say that the axioms are Euclid's except the parallel axiom. What he does in fact amounts to taking Hilbert's system, but as all readers can not be presumed to be familiar with this, he puts the matter somewhat differently. Euclid's axioms are retained except the one about parallels, but they are pieced out by assuming the first congruence theorem for triangles, by Hilbert's axiom that a given line segment may be extended a given amount in either direction, Pasch's axiom that a line in the plane of a triangle which passes between two vertices, and does not pass between a second pair nor go through a vertex, must pass between the third pair, and lastly an elaborate continuity axiom for straight lines which, incredible dictu, is never explicitly used. All this is done in a very few pages and completed by certain new constructions for perpendiculars and bisectors. The remainder of the first two chapters is of a historical nature, and follows conventional lines.

The third chapter, dealing with hyperbolic plane geometry, is the strongest in the book. The author takes a legitimate pride in the fact that he nowhere uses continuity in this chapter, but builds with not a little skill on the remaining

axioms. No construction is ever used whose possibility and correctness has not previously been shown. He makes good use of the formulas which connect the parts of a right triangle with those of a quadrilateral with three right angles, formulas which are fundamental in problems of construction in the hyperbolic plane, but which are given scant notice in many of the text-books. The only adverse comment we feel inclined to make is that in spots the logical structure is so very delicate, that an inattentive reader might suspect the existence of mistakes that are not there. The definition of "equidistant curve" on page 83 is incorrect, as it gives only half of such a curve. And thus we reach page 91 and the end of Chapter III and the middle of the book. It would have been a far more flawless piece of work if the author had written "Finis" at that point.

The fourth chapter deals with hyperbolic trigonometry. The author, contrary to the custom of certain other elementary writers, develops this by two-dimensional methods only, the fairy godmother that smoothes out all obstacles being the limiting curve or orocycle. This has been defined correctly on page 80 and on the following page we are told accurately what is meant by congruent orocyclic arcs. From this definition we may safely infer the meaning of a rational ratio of two such arcs "Right and jest, jest and right," as the immortal Disko Troop remarked. But we know nothing of incommensurable arcs on orocycles, nor do we yet know that an orocycle is a continuous curve, and when we read on page 93 "if the arcs are incommensurable we reach the same conclusion by proceeding to the limit," we feel as if we had been rudely awakened by the whole logical structure crashing down about our ears. The same seductive process of proceeding to the limit is used again on pages 128 and 139. It is so fatally easy! But why give at the outset an elaborate continuity axiom and why employ the logical rigor of the strictest sect of the pharisees during the whole first part of the book, if one is coming to this at the last? A similar lapse occurs in the handling of the equation*

$$\tanh a = \cos f(\alpha).$$

It is shown that f decreases as a increases, and it is assumed without further ado that it is continuous and differentiable!

* Pp. 106ff.

Chapter V deals with the differential of length in the hyperbolic plane, and measures of area. An area is taken as a primary concept and not further defined.

Chapter VI brings us at last to the elliptic geometry. It is brief and follows conventional methods. There is a slight slip on page 132, where the author makes use of the theorem that an exterior angle of a triangle is greater than either opposite interior angle. This is only true in the elliptic plane if the region be sufficiently small. On page 133, line 3, *FB* should read *FP*. This is the only printer's error which we have noticed in the book.

Chapter VII deals with the elliptic plane trigonometry, and is the most difficult chapter in the work. The method followed is that originally devised by Gérard, although in spots, as on page 140, the present author omits some tedious if essential details. The whole treatment suggests a didactic question of not a little interest to all who undertake to teach non-euclidean geometry. In developing the elementary parts of the subject, one may follow one of two different methods. The first is to develop a general geometry as far as possible, and to give the theorems characteristic of the particular geometries only after the general theorems have all been put in evidence. Similarly in trigonometry, a general set of formulas is derived suitable to all three classical geometries, and the distinction of one from the other depends on the value of the space constant. The second method consists in making full use in the case of each geometry of the features characteristic of that geometry. Limiting ourselves to the consideration of recent text-books we may say that Killing* and the reviewer† have followed the first method, while not only the present author, but Liebmann,‡ Manning,§ and Sommerville|| have followed the second. It is largely a question of ideal. The first method lays emphasis upon the points of similarity of the three geometries, the second emphasizes their points of distinction. The first method is shorter, as the fundamental equations have to be deduced but once, the second approximates more closely to an ideal which the late Gaston Darboux once explained to the reviewer in about these words:

* Grundlagen der Geometrie, Paderborn, 1893, especially pp. 80ff.
† The Elements of Non-Euclidean Geometry, Oxford, 1909.
‡ Nicht-euklidische Geometrie, Second Ed., Leipzig, 1912.
§ Non-Euclidean Geometry, Second Ed., Boston, 1915.
|| The Elements of Non-Euclidean Geometry, London, 1914.

"It is the distinctive characteristic of geometry, not to have a general method, but to find in the problem itself the methods best suited to its solution."

Let us note in passing an unintentional pleasantry on page 143 where we read "SOs is acute."

Our book closes with a peculiarly interesting chapter, No. 8. Most writers on non-euclidean geometry feel the necessity of showing that the subject will really "work" by exhibiting examples of one or more geometrical systems which obey just the desired hypotheses. This is usually done by a discussion of the geometry on certain surfaces of constant curvature, but this procedure is rather blind to a reader who does not know differential geometry. The present writer departs from this precedent, and gives a bird's-eye view of the three geometries at once by building the geometry of what is called "nominal lines" and "nominal points." A nominal line is nothing more nor less than a euclidean circle with regard to which a chosen fixed point has a preassigned power, or, in the limiting case, a line through that point. The credit for that idea is given to Poincaré, "Science et Hypothèse," and the reviewer, for one, had always supposed until recently that the idea was entirely derived from the lamented French geometer. As matter of fact a good share of the credit should be given to an earlier and less-known writer, DePaolis, who exhibited in 1878 a one-to-two conformal transformation from the non-euclidean to the euclidean plane, where non-euclidean lines passed over into euclidean circles orthogonal to a fixed circle, the square of whose radius was positive, negative, or zero.* The difference between this and the Poincaré scheme is largely a question of phraseology. Let us explain briefly how the plan works.

We take a fixed point O and a fixed number k, positive, negative, or zero. We define as a nominal line a line through O or a circle with regard to which O has the power k. If k be positive, we take a fixed circle with center O and radius $\sqrt{k}$; the nominal lines all cut it orthogonally, and two points determine a nominal line, unless they be inverse in the fixed circle. All points within, or on, this circle are defined as nominal points, and two nominal points will always determine just one nominal line; we have an excellent example of the

* "La trasformazione piana doppia, etc.," *Memorie della R. Accademia dei Lincei*, series 3, vol. 2 (1878).

hyperbolic plane. If k be negative, the nominal lines are circles which meet a fixed circle with center 0 and radius $\sqrt{-k}$ in diametrically opposite points, or, in the limiting case, lines through O. Two points will determine a nominal line, unless one be the reflection in O of the inverse of the other in this circle. We take as nominal points the points within the circle, and the pairs of diametrically opposite points on the circle; we have an admirable elliptic plane. Lastly, in the case where k is zero we take as nominal lines the lines or circles through O, and as nominal points the totality of finite points except O. This gives a good example of the euclidean plane. With regard to distances, we may follow the author and define them Cayley fashion by the logarithms of certain cross ratios, but this involves rather higher mathematical considerations that have been introduced before, and breaks down entirely for the euclidean case. The author's treatment of this last case is simply lamentable. He says:*

We define the nominal length of a nominal line as the length of the rectilinear segment to which it corresponds . . . the nominal length of a nominal segment is unaltered by inversion with regard to a circle of the system.

What this definition means, we do not know, as there is no indication of just how a nominal line *corresponds* to a rectilinear segment. The conclusion would seem to be that the nominal length of a nominal line is defined as the length of either the arc or the chord of the circle which is that nominal line. But a moment's thought shows that neither of these is invariant under the inversions in question. This erroneous statement suggests, however, another way of putting the thing which covers all three cases.

Let two nominal segments be defined as "congruent," if they may be transformed into one another by a succession of inversions or reflections in nominal lines. It is easy to see that no segment is congruent to a part of itself. If a segment be split in two, we may define the process of addition by saying that the length of the sum shall be the sum of their lengths. It can then be shown how any length can be measured in terms of any other.

Let us point out, in conclusion, that the whole scheme of nominal lines, beautiful as it is, represents a change on the

* P. 158.

author's part from the second to the first of those ideals in non-euclidean geometry which we dwelt on above.

J. L. COOLIDGE.

CORRECTION.

SPEAKING of M. Nau's translation of Sebokht's reference to the Hindu-Arabic numerals, in the BULLETIN for May, 1917 (volume 23, page 366), I remarked that no report of the matter "seems as yet to have appeared in English." My attention has since been called to the fact that Professor L. C. Karpinski announced the discovery in *Science* for June 21, 1912. While this does not concern Mr. Ginsburg's valuable note on the work and influence of Sebokht, the correction as to the publication of the extract in English should be made.

DAVID EUGENE SMITH.

NOTES.

THE June number (volume 18, number 4) of the *Annals of Mathematics* contains the following papers: "Fermat's last theorem and the origin and nature of the theory of algebraic numbers," by L. E. DICKSON; "The modified remainders obtained in finding the highest common factor of two polynomials," by A. J. PELL and R. L. GORDON; "Nomograms of adjustment," by L. I. HEWES; "Closed algebraic correspondences," by A. A. BENNETT; "The intersections of a straight line and hyperquadric," by J. L. COOLIDGE; "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivatives," by W. B. FITE; "Conjugate planar nets with equal invariants," by L. P. EISENHART.

AT the meeting of the Edinburgh mathematical society on May 11 the following papers were read: By L. R. FORD: "A geometrical proof of a theorem by Hurwitz and Borel"; by D. G. TAYLOR: "Geometrical illustrations of cyclant substitutions."

AT the meeting of the London mathematical society held on May 3 the following papers were read: By G. H. HARDY, "Sir George Stokes and the theory of uniform convergence"; by W. P. MILNE, "A symmetrical condition for ω-apolar triads on a cubic curve."

UNIVERSITY OF PARIS.—The following courses in mathematics are being given during the present semester (March 1–July 15).—By Professor L. C. GUICHARD: General principles of analytic geometry with applications to the theory of cyclides and correlative surfaces, two hours.—By Professor E. PICARD: Theory of integral equations with applications to problems in mathematical physics, two hours.—By Professor E. GOURSAT: Differential and integral calculus, two hours.—By Professor P. PAINLEVÉ: Rational mechanics, two hours.—By Professor L. VESSIOT: Formal resolution of the problem of three bodies by means of trigonometric series, and periodic solutions, two hours.—By Professor C. J. DE LA VALLÉE-POUSSIN: Conferences in higher analysis, two hours.

PROFESSOR E. PICARD was unanimously elected permanent secretary of the Paris Academy of sciences on April 2, as successor to the late Professor G. DARBOUX. Professor Picard became a member of the academy in 1889, succeeding Halphen.

PROFESSOR A. SPEISER has been appointed professor of mathematics in the University of Zurich.

AT the Massachusetts Institute of Technology, Dr. JOSEPH LIPKA has been promoted to an assistant professorship of mathematics.

AT the University of Nebraska, Mr. A. BABBETT and Miss LULU RUNGE have been promoted to assistant professorships of mathematics.

AT Teachers College, Columbia University, assistant professor C. B. UPTON has been promoted to an associate professorship of mathematics.

DR. L. S. DEDERICK, of Princeton University, has been appointed instructor in mathematics in the U. S. Naval Academy.

BOOK CATALOGUES:—Longmans, Green, and Company, Fourth Avenue and 30th Street, New York, text-books and reference works for colleges and technical schools, 16 pp.—The Macmillan Company, 66 Fifth Avenue, New York, books on mathematics and astronomy, 68 pp.—Galloway and Porter, Cambridge, England, short lists of mathematical and physical books, 77 entries.—Librairie Paul Ritti, 76 Avenue du Maine, Paris, Bulletin périodique, mathématique, 41 entries.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BLICHFELDT (H. F.). Finite collineation groups. Chicago, University of Chicago Press, 1917. 12mo. 12+194 pp. $1.50

DARBOUX (G.). Principes de géométrie analytique. Paris, Gauthier-Villars, 1917. 8vo. 6+520 pp. Fr. 20.00

FERNÁNDEZ BAÑOS (O.). Estudio sintético de los espacios complejos de n dimensiones. (Junta para amplición de estudios e investigaciones cientifícas, Laboratorio y seminario matemático, Tomo 2, Memoria 1.) Madrid, 1917. 8vo. 80 pp.

HUDSON (R. G.) and LIPKA (J.). A manual of mathematics. New York, Wiley, 1917. 8vo. 2+132 pp. $1.00

——. A table of integrals. New York, Wiley, 1917. 8vo. 25 pp. Paper. $0.15

LIPKA (J.). See HUDSON (R. G.).

MICHEL (C.). Cours d'algèbre et d'analyse, à l'usage de la classe de mathématiques spéciales et des candidats aux écoles du gouvernement. Paris, Alcan, 1917. 8vo. Fr. 18.00

PRINGSHEIM (A.). Vorlesungen über Zahlen- und Funktionenlehre. 1ter Band, 2te Abteilung: Unendliche Reihen mit reellen Gliedern. Leipzig, Teubner, 1916. 8vo. 222 pp. M. 10.80

II. ELEMENTARY MATHEMATICS.

AUSTIN (F. E.). Preliminary mathematics. Hanover, N. H., F. E. Austin, 1917. 169 pp. $1.20

BELL (A. H.). Algebra, theoretical and applied: a class-book for secondary, higher elementary and technical schools. London, Blackie, 1916. 8vo. 354 pp. 4s. 6d.

CONCINA (U.) e NEPPI-MODONA (A.). Trigonometria piana ad uso dei licei. Con 50 figure e 809 esercizi e problemi. Torino, Casa G. B. Petrini di Giovanni Galizio, 1917.

——. Trigonometria piana e sferica ad uso degli istituti tecnici. Con 75 figure e 1000 esercizi e problemi. Torino, Casa G. B. Petrini di Giovanni Galizio, 1917.

COULOM (F.) et WEILL (—.). Nouveau cours de géométrie théorique et pratique. 3e année. Paris, Delagrave, 1917. 12mo. Fr. 2.25

CRONFELT (S.). Lærebog i Aritmetik og Algebra (med et Tillæg om Effekter og Veksler) for Realklassen. København, Steen Hasselbalch (Jacob Erslev), 1916. 8vo. 104 pp. Kr. 1.75

FRATTINI (—.). Lezioni de algebra, geometria e trigonometria piana e sferica con molti esempi sull'intero programma del secondo bienno degl'istituti tecnici (sezione fisico-matematica). Volume 2 per la quarta classe. 3a edizione. Torino, Paravia, 1917.

GONGGRIJP (B.). Logarithmische en goniometrische tafels en bijtafels. Groningen, Noordhoff, 1916. 8vo. 230 pp. Fl. 1.70

HAWKES (H. E.), LUBY (W. A.) and TOUTON (F. C.). First course in algebra. Revised edition. Boston, Ginn, 1917. 10+301 pp. $1.00

LEMOINE (A.). Arithmétique. Cours supérieur et complémentaire. Brevet élémentaire. Livre du maître. Paris, Hachette, 1917. 12 mo. Fr. 4.00

LUBY (W. A.), See HAWKES (H. E.).

MARTINI-ZUCCAGNI (A.). Guida pratica per la risoluzione delle equazioni di 1° e 2° grado con 214 esercizi risoluti. 5a edizione. Livorno, Giusti, 1917. L. 1.50

NEPPI-MODONA (A.). See CONCINA (U.).

SMITH (O. A.). Praktische Regneopgaver for Realklassen. København, Gjellerup, 1916. 24 pp.

TOUTON (F. C.). See HAWKES (H. E.).

WEILL (—.). See COULOM (F.).

WIJDENES (P.). Logarithmen en rententafels. Groningen, Noordhoff, 1916. 8vo. 64 pp. Fl. 0.50

III. APPLIED MATHEMATICS.

BOTTASSO (M.). Analyse vectorielle générale. Volume 4: Astatique. Avec une préface de R. Marcolongo. Pavia, Mattei, 1915. 8vo. 16+159 pp. Fr. 6.00

BRUNT (D.). The combination of observations. Cambridge, University Press, 1917. 8vo. 10+219 pp. 8s.

BURTON (E. F.). The physical properties of colloidal solutions. New York, Longmans, 1916. 7+200 pp. $1.80

CREAGER (W. P.). Engineering for masonry dams. New York, Wiley, 1917. 250 pp. $2.50

DALE (R. B.). Drawing for builders. New York, Wiley, 1916. 200 pp. $1.50

DOYÈRE (C.). Cours pratique de construction navale. 1re partie: Géométrie du navire. Calculs de déplacement et de stabilité. 2e édition, revue et augmentée. Paris, Challamel, 1917. 8vo. Fr. 10.00

DUCHÊNE (E. A.). Flight without formulae: simple discussions on the mechanics of the aeroplane. Translated from the French by J. H. Ledeboer. 2d edition. New York, Longmans, 1917. 8+211 pp. $2.50

GROSSMANN (M.). Elemente der darstellenden Geometrie. (Teubners Leitfaden für den mathematischen und technischen Hochschulunterricht.) Leipzig, Teubner, 1917. 84 pp. M. 2.00

HENDERSON (W. D.). Problems in physics. New York, McGraw-Hill, 1916. 8+205 pp. $1.50

HUDSON (R. G.), LIPKA (J.), LUTHER (H. B.) and PEABODY (D.). The engineers' manual. New York, Wiley, 1917. 310 pp. $2.00

ISENTHAL (A.). See LILIENTHAL (O.).

JAHNKE (E.). See MEHMKE (R.).

JUDGE (A. W.). The properties of aerofoils and aerodynamic bodies. London, Whittaker, 1917. 8vo. 10+298 pp. 15s.

LEDEBOER (J. H.). See DUCHÊNE (E. A.).

LILIENTHAL (G.). See LILIENTHAL (O.).

LILIENTHAL (O.). Bird flight as the basis of aviation: a contribution toward a system of aviation. With a biographical introduction and addendum by G. Lilienthal. Translated from the second edition by A. Isenthal. New York, Longmans, 1917. 8vo. 14+142 pp. $2.50

LIPKA (J.). See HUDSON (R. G.).

LUTHER (H. B.). See HUDSON (R. G.).

MARCOLONGO (R.). See BOTTASSO (M.).

MEHMKE (R.). Leitfaden zum graphischen Rechnen. (Sammlung mathematisch-physikalischer Lehrbücher, herausgegeben von E. Jahnke.) Leipzig, Teubner, 1917. 8+152 pp. M. 4.80

MELAN (J.) and STEINMAN (D. B.). Plain and reinforced concrete arches. 2d edition. New York, Wiley, 1917. 171 pp. $2.00

NOTTAGE (W. H.). The calculation and measurement of inductance and capacity. London, The Wireless Press, 1917. 8vo. 137 pp. 2s. 6d.

PEABODY (D.). See HUDSON (R. G.).

RICCARDO-BRAUZZI (—.). Cours d'aéronautique générale. 1re partie: Etude des questions qui intéressent tous les modes de locomotion aérienne. Paris, Dunod et Pinat, 1917. 8vo. 344 pp. Fr. 30.00

RICHARDSON (O. W.). The emission of electricity from hot bodies. New York, Longmans, 1916. 6+304 pp. $2.75

RUGGLES (C. L.). Stresses in wire-wrapped guns and in gun carriages. New York, Wiley, 1916. 270 pp. $3.00

SERVICE des grands forces hydrauliques (Région des Alpes). Tome VII: Compte rendu des résultats des études et travaux au 31. décembre 1915. Paris, Dunod et Pinat, 1917. 8vo. 481 pp. Fr. 30.00

STEINMAN (D. B.). See MELAN (J.).

WATERBURY (L. A.). Stresses in structural steel angles. New York, Wiley, 1917. 82 pp. $1.25

TWENTY-SIXTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

ALEXANDER, J. W., II. On the factorization of Cremona plane transformations. Read Dec. 28, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 3, pp. 295–300; July, 1916.

BATEMAN, H. On systems of partial differential equations and the transformation of spherical harmonics. Read Feb. 26, 1916. *Proceedings of the Royal Society of Edinburgh*, vol. 36, Nos. 3–4, pp. 300–312; March, 1917.

BEAL, F. W. A congruence of circles. Read April 24, 1915. *Annals of Mathematics*, ser. 2, vol. 17, No. 4, pp. 180–187; June, 1916.

BEATTY, S. The inversion of an analytic function. Read April 28, 1917. *Bulletin of the American Mathematical Society*, vol. 23, No. 8, pp. 347–353; May, 1917.

BENNETT, A. A. A case of iteration in several variables. Read April 29, 1916. *Annals of Mathematics*, ser. 2, vol. 17, No. 4, pp. 188–196; June, 1916.

—— An existence theorem for the solution of a type of real mixed difference equation. Read April 29, 1916. *Annals of Mathematics*, ser. 2, vol. 18, No. 1, pp. 24–30; Sept., 1916.

—— Newton's method in general analysis. Read (Southwestern Section) Dec. 2, 1916. *Proceedings of the National Academy of Sciences*, vol. 2, No. 10, pp. 592–598; Oct., 1916.

—— Closed algebraic correspondences. Read (Chicago) Dec. 22, 1916. *Annals of Mathematics*, ser. 2, vol. 18, No. 4, pp. 200–208; June, 1917.

BIRKHOFF, G. D. Infinite products of analytic matrices. Read Dec. 27, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 3, pp. 386–404; July, 1916.

—— Dynamical systems with two degrees of freedom. Read Dec. 27, 1915 and Sept. 5, 1916. *Proceedings of the National Academy of Sciences*, vol. 3, No. 4, pp. 314–316; April, 1917. *Transactions of the American Mathematical Society*, vol. 18, No. 2, pp. 199–300; April, 1917.

BLUMBERG, H. On the factorization of expressions of various types. Read (Southwestern Section) Nov. 28, 1914 and (Chicago) Dec. 29, 1914. *Transactions of the American Mathematical Society*, vol. 17, No. 4, pp. 517–544; Oct., 1916.

—— Certain general properties of functions. Read (Chicago) April 10, 1914 and (Chicago) Dec. 22, 1916. *Proceedings of the National Academy of Sciences*, vol. 2, No. 11, pp. 646–649; Nov., 1916. *Annals of Mathematics*, ser. 2, vol. 18, No. 3, pp. 147–160; March, 1917.

BÔCHER, M. La méthode des approximations successives pour les systèmes différentiels. Read Sept. 8, 1914. Section 22, pp. 112–116, of Leçons sur les Méthodes de Sturm dans la Théorie des Equations différentielles linéaires et leurs Développements modernes. (Collection de Monographies sur la Théorie des Fonctions publiée sous la Direction de M. Émile Borel.) Paris, Gauthier-Villars, 1917.

BORGER, R. L. A theorem in the analysis of real variables. Read (Chicago) April 21, 1916. *Bulletin of the American Mathematical Society*, vol. 23, No. 6, pp. 287–290; March, 1917.

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BUCHANAN, D. Oscillations near an isosceles-triangle solution as the finite bodies become unequal. Read (Chicago) Dec. 28, 1914. *American Journal of Mathematics*, vol. 39, No. 1, pp. 41–58; Jan., 1917.

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BURGESS, R. W. The comparison of a certain case of the elastic curve with its approximation. Read Sept. 4, 1916. *Physical Review*, ser. 2, vol. 9, No. 3, pp. 193–197; March, 1917.

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—— Examples of a remarkable class of series. Read April 28, 1917. *Bulletin of the American Mathematical Society*, vol. 23, No. 9, pp. 407–425; June, 1917.

CARPENTER, A. F. Ruled surfaces whose flecnode curves have plane branches. Read (San Francisco) May 22, 1914, (Chicago) Dec. 28, 1914, and (Chicago) April 2, 1915. *Transactions of the American Mathematical Society*, vol. 16, No. 4, pp. 509–532; Oct., 1915.

CARSLAW, H. S. A trigonometrical sum and the Gibbs' phenomenon in Fourier's series. Read Oct. 30, 1915. *American Journal of Mathematics*, vol. 39, No. 2, pp. 185–197; April, 1917.

CHITTENDEN, E. W. On the equivalence of écart and voisinage. Read (Chicago) April 21, 1916. *Transactions of the American Mathematical Society*, vol. 18, No. 2, pp. 161–166; April, 1917.

COBLE, A. B. Point sets and allied Cremona groups (Part II). Read April 25, 1914. *Transactions of the American Mathematical Society*, vol. 17, No. 3, pp. 345–385; July, 1916.

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EISENHART, L. P. Sulle superficie di rotolamento e le trasformazioni di Ribaucour. Read Aug. 4, 1915. *Accademia dei Lincei, Rendiconti*, vol. 24, 2d sem., No. 8, pp. 349–352; Nov. 5, 1915.

—— Surfaces generated by the motion of an invariable curve whose points describe straight lines. Read Dec. 28, 1915. *Rendiconti del Circolo Matematico di Palermo*, vol. 41, No. 1, pp. 94–102; Jan.–April, 1916.

—— Conjugate systems with equal point invariants. Read April 29, 1916. *Annals of Mathematics*, ser. 2, vol. 18, No. 1, pp. 7–17; Sept., 1916.

—— Deformable transformations of Ribaucour. Read Sept. 4, 1916. *Transactions of the American Mathematical Society*, vol. 17, No. 4, pp. 437–458; Oct., 1916.

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